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Equivalent classes of integrable non-linear evolution equations and generalised Miura transformations

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Abstract. We write down the recursion operator which defines the class of integrable non-linear evolution equations (NEE) associated with a spectral problem (SP) for matrices of rank 2 depending quadratically on the spectral parameter. Then we show the existence of four different transformations which transform the given SP into the well known one of Zakharov and Shabat (ZS); by composing these transformations we recover, among others, the so-called elementary Bäcklund transformations for the ZS SP. Under some restrictions, one is able to prove the complete equivalence of the given class of NEE to the one associated with the ZS SP.

1. Introduction

A few years ago one of the authors (Levi 1981) introduced a new spectral problem (SP)

$$\psi_x = \begin{pmatrix} -q & \lambda q \\ -\lambda & \lambda^2 + r \end{pmatrix} \psi = U(q, r; \lambda)\psi \tag{1.1}$$

where λ is a complex parameter, (r, q) are two (x, t)-dependent fields, $\psi = \psi(x, t; \lambda)$ is a matrix wavefunction depending parametrically on t and, by a subscript, we mean partial differentiation. For this SP the generalised Volterra equation (Wadati 1976) for an infinite number of interacting predator-prey species appears as a Bäcklund transformation (BT).

Levi in 1981 has been able, using the Wronskian technique (Calogero 1976) to obtain a hierarchy of non-linear evolution equations (NEE) and BT when the fields $(q(x, t), r(x, t))$ go asymptotically to a constant value. Later on Levi *et al* (1984a), considering the NEE associated with the SP (1.1) for $(q(x, t), r(x, t))$ vanishing asymptotically, have shown that one could introduce a transformation matrix T which mapped the SP (1.1) into the well known Zakharov-Shabat (ZS) SP (Zakharov and Shabat 1972)

$$\phi_x = \begin{pmatrix} -i\mu & u \\ v & i\mu \end{pmatrix} \phi = W(u, v; \mu)\phi \tag{1.2}$$

where μ is a spectral parameter, (u, v) are two (x, t)-dependent fields and $\phi = \phi(x, t; \mu)$ is a matrix wavefunction. However, they could not find the recursion operator nor

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did they try to prove, using the known transformation T, the equivalence of the hierarchies one could construct starting from the two given SP.

In a recent report, Li *et al* (1986) have given five different classes of NEE associated with different SP described by matrices of rank 2 and linear in the spectral parameter such that all are gauge equivalent to the SP (1.2). So it seemed interesting to prove this equivalence in this case as well, when the two SP have a different dependence from the spectral parameter.

In this paper, we follow the well known procedure introduced by Ablowitz *et al* (1974) for the SP (1.2) and construct in § 2 the recursion operator associated with the SP (1.1) for vanishing potentials. Section 3 is devoted to the presentation of the four different T matrices which transform ϕ into ψ. In correspondence with them we get the relation between the corresponding spectral parameters (μ, λ) and fields $((u, v), (q, r))$, i.e. the generalised Miura transformations (GMT). By composing the GMT we are able to get a set of BT for the SP (1.1) and for the SP (1.2), among which one finds the elementary BT (EBT) (Calogero and Degasperis 1984). We use then the GMT to get new solutions to the NEE associated with the SP (1.1).

Finally, in § 4 we prove, under some restrictive assumptions on the relation between λ and μ, the complete equivalence of the two classes of NEE associated with the SP (1.1) and (1.2) following the procedure introduced by Li *et al* (1986).

2. Construction of the recursion operator for the SP (1.1)

Given the SP (1.1), the associated NEE are obtained by requiring the existence of a denumerable set of matrices V^m, $m = 1, 2, \ldots$, such that

$$\psi_t = V^m(q, r; \lambda)\psi \tag{2.1}$$

and are given by

$$U_t - V^m_x + [U, V^m] = 0 \tag{2.2}$$

where by $[A, B]$ we mean the usual matrix commutator $AB - BA$.

Given the specific structure of U as a function of λ (see (1.1)), we can propose the following form for V^m:

$$V^m = \sum_{j=0}^{m-1} \begin{pmatrix} a^{2j} & 0 \\ 0 & d^{2j} \end{pmatrix} \lambda^{2(m-j)} + \begin{pmatrix} \bar{a}^{2m} & 0 \\ 0 & \bar{d}^{2m} \end{pmatrix} + \sum_{j=0}^{m-1} \begin{pmatrix} 0 & b^{2j+1} \\ c^{2j+1} & 0 \end{pmatrix} \lambda^{2(m-j)-1}. \tag{2.3}$$

Substituting (2.3) and U, given by (1.1), into (2.2) and equating the coefficients of the various different powers of λ to zero, we get a set of coupled ordinary differential equations of first order for the set of coefficients $(a^{2j}, d^{2j}, b^{2j+1}, c^{2j+1}, \bar{a}^{2m}, \bar{d}^{2m})$. Noticing that at all levels $(j = 1, 2, \ldots, m-1)$ $d^{2j} = -a^{2j}$ is always compatible and defining $e^{2j} = c^{2j+1} - a^{2j}$ $(j = 0, 1, \ldots, m)$, these equations thus define the following recursive relation:

$$\begin{pmatrix} a^{2j} \\ e^{2j} \end{pmatrix} = \mathsf{L}\begin{pmatrix} a^{2(j-1)} \\ e^{2(j-1)} \end{pmatrix} + \alpha^{2j}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.4}$$

where

$$\mathsf{L} = \begin{pmatrix} -D + q - IrD & q + IqD \\ -r - IrD & D - r + IqD \end{pmatrix} \tag{2.5}$$

with α^{2j} constants. By D we denote just the operator of partial differentiation with respect to x and by I its inverse, i.e. the integral operator $I = \int^x dy$ defined on functions which go to zero somewhere on the x axis. Moreover we can set $\bar{a}^{2m} = a^{2m}$, $\bar{d}^{2m} = e^{2m}$ and thus it follows that

$$\begin{pmatrix} -q_t \\ r_t \end{pmatrix} = D\begin{pmatrix} a^{2m} \\ e^{2m} \end{pmatrix} = D\sum_{j=0}^{m-1} \alpha^{2j}(\mathbf{L})^{m-j}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.6}$$

is the whole class of NEE associated with the SP (1.1) for vanishing potentials.

The first elements of this hierarchy (2.6) are

$$q_t = 2\alpha^0(-q_{xx} - 3q_x r + 3qq_x - q^3 - 3qr^2 + 6q^2 r)_x + 2\alpha^2(q_x - q^2 + 2qr)_x - 2\alpha^4 q_x$$

$$r_t = 2\alpha^0(-r_{xx} - 3qr_x + 3rr_x - r^3 - 3q^2 r + 6qr^2)_x + 2\alpha^2(-r_x + r^2 - 2qr)_x - 2\alpha^4 r_x$$

for $m = 3$.

As far as the ZS SP (1.2) is concerned, the recursion operator and thus the associated class of NEE has been presented for the first time by Ablowitz *et al* (1974):

$$\mathbf{M} = \begin{pmatrix} -D + 2uIv & -2uIu \\ 2vIv & D - 2vIu \end{pmatrix} \tag{2.7}$$

$$\begin{pmatrix} -u_t \\ v_t \end{pmatrix} = \sum_{j=0}^{m-1} \beta^{2j}(\mathbf{M})^{(m-j)}\begin{pmatrix} -2u \\ -2v \end{pmatrix}. \tag{2.8}$$

3. GMT and their composition

Let us require the existence of a gauge transformation T between the wavefunctions ψ and ϕ of the SP (1.1) and (1.2), i.e.

$$\psi = T\phi \tag{3.1}$$

where T shall be a matrix function of rank 2 depending, in a way to be determined, on the set of fields $((q, r), (u, v))$ and linearly on the spectral parameters λ and μ. The compatibility of (3.1) with (1.1) and (1.2) gives us an equation for the matrix T:

$$T_x = U(q, r; \lambda)T - TW(u, v; \mu). \tag{3.2}$$

Under the hypothesis of a quadratic dependence of μ on λ equation (3.2) can, due to the arbitrariness of λ, be solved for T to give the following four different solutions:

$$T_1 = e^{i\mu x}\begin{pmatrix} \lambda & 0 \\ 1 & -K \end{pmatrix} \tag{3.3}$$

$$T_2 = e^{i\mu x}\begin{pmatrix} -\delta & qK \\ 0 & \lambda K \end{pmatrix} \tag{3.4}$$

for $\mu = \gamma - \frac{1}{2}i\lambda^2$, where the function K is defined by

$$K_x = (r - q - 2i\gamma)K \tag{3.5}$$

and γ and δ are arbitrary constants;

$$T_3 = e^{-i\mu x}\begin{pmatrix} 0 & \lambda \\ -\tilde{K} & 1 \end{pmatrix} \tag{3.6}$$

$$T_4 = e^{-i\mu x}\begin{pmatrix} q\tilde{K} & -\tilde{\delta} \\ \lambda\tilde{K} & 0 \end{pmatrix} \tag{3.7}$$

where now $\mu=\gamma+\frac{1}{2}\mathrm{i}\lambda^2$, the function $\tilde{K}$ is defined by $\tilde{K}_x=(r-q+2\mathrm{i}\gamma)\tilde{K}$ and $\tilde{\delta}$ is an arbitrary constant.

Corresponding to the transformation matrix (3.3) we have the following GMT:

$$u=-qK \qquad v=-r/K \tag{3.8}$$

which allows us to rewrite (3.5) in this case in the following way:

$$\xi_x=-v\xi^2+u-2\mathrm{i}\gamma\xi \qquad K=\xi. \tag{3.9}$$

Equation (3.9) is nothing but the equation defining the intermediate wavefunction ξ of the ZS SP (1.2) in terms of the matrix wavefunction

$$\phi=\begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix} \qquad \xi(x,t;k)=\frac{\varphi_{11}(x,t;k)+b\varphi_{12}(x,t;k)}{\varphi_{21}(x,t;k)+b\varphi_{22}(x,t;k)} \tag{3.10}$$

corresponding to $k=\mathrm{i}\gamma$ and where b is just an arbitrary constant depending on the normalisation condition for the function ξ (Levi *et al* 1984b). In such a way the GMT (3.8) can be inverted, obtaining (q, r) in terms of (u, v)

$$q=-u/\xi \qquad r=-v\xi. \tag{3.11}$$

This result has already been obtained by Levi *et al* (1984a). Corresponding to the transformation (3.4) we obtain

$$v=\delta/K \qquad u=(q_x+qr)K/\delta. \tag{3.12}$$

Taking into account the intermediate wavefunction ξ we can invert the GMT (3.12) to obtain

$$q=v\xi \qquad r=v\xi-v_x/v+2\mathrm{i}\gamma. \tag{3.13}$$

Corresponding to the transformation matrix (3.6) we have

$$v=-q\tilde{K} \qquad u=-r/\tilde{K}. \tag{3.14}$$

This GMT allows us, in this case, to identify $\tilde{K}$ with the inverse of ξ, i.e. $\tilde{K}=1/\xi$, and thus to obtain the inverse GMT

$$q=-v\xi \qquad r=-u/\xi. \tag{3.15}$$

Finally the transformation (3.7) corresponds to the GMT

$$u=\tilde{\delta}/\tilde{K} \qquad v=(q_x+qr)\tilde{K}/\tilde{\delta}. \tag{3.16}$$

Taking into account the intermediate wavefunction ξ, the inverse of the GMT (3.16) is

$$q=u/\xi \qquad r=u/\xi-u_x/u-2\mathrm{i}\gamma. \tag{3.17}$$

Our first comment on the GMT obtained here and their inverse is that, while equations (3.8), (3.11), (3.14) and (3.15) give asymptotically vanishing potentials from asymptotically vanishing potentials, the remaining GMT transform vanishing potentials into potentials which go asymptotically at least to a constant value. If we set $\gamma=0$ then K and $\tilde{K}$ are proportional and the number of different GMT reduces.

The transformations T_j ($j = 1, \ldots, 4$) can be combined to obtain BT for the NEE associated with the SP (1.1) and (1.2). If we write down a matrix D_{ij} such that $D_{ij} = T_i^{-1}T_j$ then the matrix D_{ij} is a Darboux matrix corresponding to a BT for the NEE associated with the SP (1.2), i.e. a transformation which allows us to pass from a set of potentials (u, v) to a new set of potentials, denoted by (u^0, v^0), while if $\tilde{D}_{ij}$ is such that $\tilde{D}_{ij} = T_iT_j^{-1}$, this is a Darboux matrix corresponding to the SP (1.1) (from (q, r) to (q^0, r^0)). It is immediate to verify that we can construct six *a priori* different Darboux matrices D_{ij} and $\tilde{D}_{ij}$ and thus six *a priori* different BT for each SP.

In appendix 1 we give the explicit expressions for the 12 different Darboux matrices with their associated BT. Here we just present the main results that can be derived from them. The Darboux matrices D_{21} and D_{43} give rise to the well known EBT for the ZS SP (1.2). These same EBT can be obtained by successive applications of the BT associated with the Darboux matrices D_{42} (resp. D_{31}) and D_{41} (resp. D_{32}) with an appropriate choice of the arbitrary constant coefficients δ and $\tilde{\delta}$. We notice that the Darboux matrices D_{31} and D_{42} correspond to an invariance property of the SP (1.2) which corresponds to identifying, up to an explicit (γx)-dependent factor, u^0 with v and v^0 with u; if moreover $\delta = \tilde{\delta}$ then the two Darboux matrices are equal. If we set $\delta = -\tilde{\delta}$ then we have the following identity: $D_{43}(u, v, u^0, v^0) = (1/\delta)D_{21}^{-1}(u^0, v^0, u, v)$ which implies that, in some way, the EBT are mutual inverses.

In all cases, apart from a multiplicative λ-dependent term, the Darboux matrices are all linear in μ but asymptotically (in μ) singular matrices. Finally we notice that, while the Darboux matrices D_{21} and D_{43} connect different solutions of the same SP (1.2), the other four Darboux matrices start from a SP (1.2) corresponding to the eigenvalue μ to go over to the SP (1.2) corresponding to the eigenvalue $\mu^0 = 2\gamma - \mu$.

As for the BT associated with the SP (1.1) it is worthwhile noticing that $\tilde{D}_{32}$, $\tilde{D}_{41}$, and $\tilde{D}_{21}$, $\tilde{D}_{43}$ give the same BT.

All the BT give explicitly the new solution in terms of the old one without the necessity of solving any differential equation. Moreover, by applying in succession $\tilde{D}_{31}$ and $\tilde{D}_{21}$ we obtain the same result as $\tilde{D}_{32}$.

The GMT (3.8) and (3.11) has been used previously (Levi *et al* 1984a) to obtain explicit solutions of the NEE associated with the SP (1.1). Now, we apply the GMT (3.12) and (3.13) to obtain new solutions for the NEE associated with the SP (1.1) in terms of those of the ZS SP. Using the knowledge of the solution of the ZS SP corresponding to N solitons over a generic background potential, as expressed in terms of the intermediate wavefunction ξ (3.10) of the background potential calculated at the position of the N poles of the transmission coefficient, the corresponding solution of the SP (1.1) can be immediately evaluated. Instead of presenting the general formula, easily reconstructed following, for example, the works of Neugebauer (see Levi *et al* 1984a and references therein), we just give the explicit formulae corresponding to a soliton over a zero background.

If we take for u and v the zero solution, (3.13) is trivial; for $u = 0$ and $v = v_0$ constant then r and q differ only by a complex constant $r - q = 2\mathrm{i}\gamma$ and

$$q = \frac{v_0}{\mathrm{i}v_0/2\gamma + c\,\mathrm{e}^{2\mathrm{i}\gamma x}}.$$

To have a real solution γ must be purely imaginary and thus q goes asymptotically on one side to zero and on the other to $-2\mathrm{i}\gamma$. For (u, v) being the one-soliton solution, the formulae for (q, r) already become quite involved (see appendix 2) but preserve the property of going asymptotically to a constant value for a generic value of γ.

4. Equivalence of the two classes of NEE

We would like to prove here that the GMT (3.8), (3.12), (3.14) and (3.16) together with their inverses (3.11), (3.13), (3.15) and (3.17) transform any solution of the NEE associated with the ZS SP (1.2) into a solution of the NEE associated with the SP (1.1). First of all it is easy to prove by direct inspection that, if we apply any of the GMT with $\gamma \neq 0$ we cannot obtain a NEE of the class (2.6) from one of the class (2.8). So, in the following, choosing $\gamma = 0$, we shall write down an operator $\mathbf{S}$ such that

$$\mathbf{S}\begin{pmatrix}1\\1\end{pmatrix} = -2\begin{pmatrix}u\\v\end{pmatrix} \tag{4.1}$$

$$\mathbf{S}I\begin{pmatrix}-q_t\\r_t\end{pmatrix} = \varepsilon\begin{pmatrix}-u_t\\v_t\end{pmatrix} \tag{4.2}$$

$$\mathbf{SL} = \varepsilon\mathbf{MS} \tag{4.3}$$

for each of the four GMT given in § 3, where ε can take the values ± 1 according to the GMT studied and $\mathbf{L}$ and $\mathbf{M}$ are the recursion operators for the two hierarchies of NEE, given respectively by (2.5) and (2.7).

Given the operator $\mathbf{S}$ it is trivial to prove that if (q, r) is a set of solutions of the NEE (2.6) then the (u, v), defined through the corresponding GMT, is a solution of the NEE (2.8) with $\alpha^{2j} = \varepsilon^{j-m+1}\beta^{2j}$ and vice versa (due to the invertibility of all the GMT and the matrices $\mathbf{S}$ studied).

For the GMT (3.8) equations (4.1)-(4.3) define

$$\mathbf{S} = \begin{pmatrix} K(q-D) & qK \\ r/K & (1/K)(r-D) \end{pmatrix}$$

with $\varepsilon = 1$; for the GMT (3.16)

$$\mathbf{S} = \begin{pmatrix} -\tilde{\delta}/\tilde{K}; & -\tilde{\delta}/\tilde{K} \\ (\tilde{K}/\tilde{\delta})(D^2 + rD - q_x - qr); & -(\tilde{K}/\tilde{\delta})(qD + q_x + qr) \end{pmatrix}$$

with $\varepsilon = -1$; for the GMT (3.14)

$$\mathbf{S} = \begin{pmatrix} r/\tilde{K}; & (1/\tilde{K})(r-D) \\ \tilde{K}(q-D); & \tilde{K}q \end{pmatrix}$$

with $\varepsilon = -1$ and finally for the GMT (3.12)

$$\mathbf{S} = \begin{pmatrix} (K/\delta)(D^2 + rD - q_x - qr); & -(K/\delta)(qD + q_x + qr) \\ -\delta/K; & -\delta/K \end{pmatrix}$$

with $\varepsilon = 1$.

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Appendix 1

In the following we just write down one after the other the 12 matrices D_{ij} and $\tilde{D}_{ij}$ together with their corresponding BT:

$$D_{21}(u, v; u^0, v^0; \lambda, \mu) = -\frac{1}{\delta\lambda}\begin{pmatrix} 2i\mu - 2i\gamma + u/\xi; & -u \\ -\delta/\xi; & \delta \end{pmatrix}$$

$$\begin{cases} u^0 = -(u_x + 2i\gamma u)/\delta + u^2 v^0/\delta^2 \\ v^0_x = 2i\gamma v^0 - u(v^0)^2/\delta + \delta v \end{cases} \qquad \mu_0 = \mu$$

$$D_{31}(u, v; u^0, v^0; \lambda, \mu) = \begin{pmatrix} 0; & e^{-2i\gamma x} \\ e^{2i\gamma x}; & 0 \end{pmatrix}$$

$$\begin{cases} u^0 = v\, e^{-4i\gamma x} \\ v^0 = u\, e^{4i\gamma x} \end{cases} \qquad \mu_0 = 2\gamma - \mu$$

$$D_{32}(u, v; u^0, v^0; \lambda, \mu) = \frac{e^{-2i\gamma x}}{\lambda}\begin{pmatrix} -\delta\xi^0 e^{4i\gamma x}; & -2i\mu + 2i\gamma - v^0\xi^0 \\ -\delta\, e^{4i\gamma x}; & -v^0 \end{pmatrix}$$

$$\begin{cases} u^0 = (v_x - 2i\gamma v)\, e^{-4i\gamma x}/\delta + v^0 v^2 e^{-8i\gamma x}/\delta^2 \\ v^0_x = 2i\gamma v^0 + v(v^0)^2 e^{-4i\gamma x}/\delta - u\delta\, e^{4i\gamma x} \end{cases} \qquad \mu_0 = 2\gamma - \mu$$

$$D_{41}(u, v; u^0, v^0; \lambda, \mu) = \frac{e^{2i\gamma x}}{\tilde{\delta}\lambda}\begin{pmatrix} \tilde{\delta}\, e^{-4i\gamma x}/\xi; & -\tilde{\delta}\, e^{-4i\gamma x} \\ -2i\mu + 2i\gamma - u/\xi; & u \end{pmatrix}$$

$$\begin{cases} u^0_x = -2i\gamma u^0 - u(u^0)^2 e^{4i\gamma x}/\delta + \tilde{\delta} v\, e^{-4i\gamma x} \\ v^0 = -(u_x + 2i\gamma u)\, e^{4i\gamma x}/\tilde{\delta} + u^2 u^0 e^{8i\gamma x}/\tilde{\delta}^2 \end{cases} \qquad \mu_0 = 2\gamma - \mu$$

$$D_{42}(u, v; u^0; v^0; \lambda, \mu) = \begin{pmatrix} 0 & e^{-2i\gamma x} \\ e^{2i\gamma x}\delta/\tilde{\delta} & 0 \end{pmatrix}$$

$$\begin{cases} u^0 = v(\tilde{\delta}/\delta)\, e^{-4i\gamma x} \\ v^0 = u(\delta/\tilde{\delta})\, e^{4i\gamma x} \end{cases} \qquad \mu_0 = 2\gamma - \mu$$

$$D_{43}(u, v; u^0, v^0; \lambda, \mu) = \frac{1}{\tilde{\delta}\lambda}\begin{pmatrix} -\tilde{\delta}; & \tilde{\delta}\xi \\ v; & 2i\mu - 2i\gamma - v\xi \end{pmatrix}$$

$$\begin{cases} u^0_x = -2i\gamma u^0 - v(u^0)^2/\tilde{\delta} + \tilde{\delta} u \\ v^0 = -(v_x - 2i\gamma v)/\tilde{\delta} + u^0 v^2/\tilde{\delta}^2 \end{cases} \qquad \mu_0 = \mu$$

$$\tilde{D}_{21}(q, r; q^0, r^0; \lambda) = \frac{-1}{\lambda}\begin{pmatrix} \delta - q(K^0/K) & \lambda q(K^0/K) \\ -\lambda(K^0/K) & \lambda^2(K^0/K) \end{pmatrix}$$

$$\begin{cases} q^0 = -r \\ r^0 = -q - r_x/r \end{cases} \qquad \lambda^0 = \lambda$$

$$\tilde{D}_{31}(q, r; q^0, r^0; \lambda) = -\frac{\exp[-2i(\gamma - i\lambda^2/2)x]}{K\lambda}\begin{pmatrix} -i\lambda; & i\lambda^2 \\ 0; & \lambda \end{pmatrix}$$

$$\begin{cases} q^0 = r \\ r^0 = q \end{cases} \qquad \lambda^0 = i\lambda$$

$$\tilde{D}_{32}(q, r; q^0, r^0; \lambda) = -\frac{\exp[-2i(\gamma - i\lambda^2/2)x]}{\delta\lambda K}\begin{pmatrix} 0; & -i\lambda\delta \\ -\lambda KK^0; & qKK^0 - \delta \end{pmatrix}$$

$$\begin{cases} q^0 = -q \\ r^0 = -r - q_x/q \end{cases} \qquad \lambda^0 = i\lambda$$

$$\tilde{D}_{41}(q, r; q^0, r^0; \lambda) = -\frac{\exp[-2i(\gamma - i\lambda^2/2)x]}{\lambda K}\begin{pmatrix} \tilde{\delta} - q^0 KK^0; & -\delta\lambda \\ -i\lambda KK^0; & 0 \end{pmatrix}$$

$$\begin{cases} q^0 = -q \\ r^0 = -r - q_x/q \end{cases} \qquad \lambda^0 = i\lambda$$

$$\tilde{D}_{42}(q, r; q^0, r^0; \lambda) = -\frac{\exp[-2i(\gamma - i\lambda^2/2)x]}{\delta\lambda K}\begin{pmatrix} \lambda q^0 KK^0; & \delta\tilde{\delta} - qq^0 KK^0 \\ i\lambda^2 KK^0; & -i\lambda qKK^0 \end{pmatrix}$$

$$\begin{cases} q^0 = -q_x/q - r \\ r^0 = q - q_x^0/q^0 \end{cases} \qquad \lambda^0 = i\lambda$$

$$\tilde{D}_{43}(q, r; q^0, r^0; \lambda) = -\frac{1}{\lambda}\begin{pmatrix} \tilde{\delta} - q^0(\tilde{K}^0/\tilde{K}); & \lambda q^0(\tilde{K}^0/\tilde{K}) \\ -\lambda(\tilde{K}^0/\tilde{K}); & \lambda^2(\tilde{K}^0/\tilde{K}) \end{pmatrix}$$

$$\begin{cases} q^0 = -r \\ r^0 = -q - r_x/r \end{cases} \qquad \lambda^0 = \lambda.$$

Appendix 2

The one-soliton solution of the ZS SP is

$$u(x, t) = -\frac{2\bar{\nu}(t)\exp(-2i\lambda_2 x)}{D(x, t)}$$

$$v(x, t) = \frac{2\nu(t)\exp(2i\lambda_1 x)}{D(x, t)}$$

where, if the solution is to be bounded, we have to require $\mathrm{Im}\,\lambda_1 > 0$, $\mathrm{Im}\,\lambda_2 < 0$; ν, $\bar{\nu}$, ω are functions whose time dependence depends on the NEE in the hierarchy one is choosing and

$$D(x, t) = 1 - \frac{\nu(t)\bar{\nu}(t)}{(\lambda_1 - \lambda_2)^2}\exp[2i(\lambda_1 - \lambda_2)x].$$

The corresponding intermediate wavefunction ξ is given by

$$\xi = -i\frac{\omega(t)\exp(-2i\gamma x)[(\lambda_2 - \gamma)D(x, t) + \lambda_1 - \lambda_2] - \bar{\nu}(t)\exp(-2i\lambda_2 x)}{\lambda_2 - \lambda_1 + (\lambda_1 - \gamma)D(x, t) + \omega(t)\nu(t)\exp[2i(\lambda_1 - \gamma)x]}$$

and thus

$$r(x, t) = 2q(x, t) - 2\mathrm{i}\frac{D(x, t)(\lambda_2 - \gamma)(\lambda_1 - \gamma)}{\lambda_2 - \lambda_1 + (\lambda_1 - \gamma)D(x, t) + \omega(t)\nu(t)\exp[2\mathrm{i}(\lambda_1 - \gamma)x]}$$

$$q(x, t) = -\frac{2\mathrm{i}}{D}\left(\frac{\nu(t)\omega(t)\exp[2\mathrm{i}(\lambda_1 - \gamma)x][\lambda_1 - \lambda_2 + (\lambda_2 - \gamma)D(x, t)] - (1 - D)(\lambda_1 - \lambda_2)^2}{\lambda_2 - \lambda_1 + (\lambda_1 - \gamma)D(x, t) + \omega(t)\nu(t)\exp[2\mathrm{i}(\lambda_1 - \gamma)x]}\right).$$

$q(x, t)$ shall be vanishing asymptotically only if $\mathrm{Im}\,\lambda_2 < \mathrm{Im}\,\gamma < \mathrm{Im}\,\lambda_1$ and r is going asymptotically to constant values.

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